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A Dirichlet-type integral on spheres, applied to the fluid/gravity correspondence

David D. K. Chow

George P. & Cynthia W. Mitchell Institute for Fundamental Physics & Astronomy, Texas A&M University, College Station, TX 77843-4242, USA

chow@physics.tamu.edu

Abstract

We evaluate an analogue of an integral of Dirichlet over the sphere S^D, but with an integrand that is independent of $\lfloor (D+1)/2 \rfloor$ Killing coordinates. As an application, we evaluate an integral that arises when comparing a conformal fluid on S^D and black holes in $(D+2)$-dimensional anti-de Sitter spacetime.

1 Introduction

There is a class of functions that are particularly easy to integrate over the n-dimensional unit sphere S^n, namely monomials in the cartesian coordinates for $\mathbb{R}^{n+1} \supset S^n$. Let x_i, $i = 1, \dots, n+1$ be such coordinates, so S^n is the hypersurface $\sum_{i=1}^{n+1} x_i^2 = 1$. A well-known result of Dirichlet [1] is that, for non-negative integers α_j,

$$\int_{S^n} \prod_{j=1}^{n+1} x_j^{\alpha_j} = \begin{cases} 0, & \text{some } \alpha_j \text{ is odd,} \\ \dfrac{2\prod_{j=1}^{n+1}\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_j)}{\Gamma(\frac{n+1}{2}+\frac{1}{2}\sum_{i=1}^{n+1}\alpha_i)}, & \text{all } \alpha_j \text{ are even.} \end{cases} \tag{1.1}$$

More generally, we have, for any real and non-negative α_j,

$$\int_{S^n} \prod_{j=1}^{n+1} |x_j|^{\alpha_j} = \frac{2\prod_{j=1}^{n+1}\Gamma(\frac{1}{2}+\frac{1}{2}\alpha_j)}{\Gamma(\frac{n+1}{2}+\frac{1}{2}\sum_{i=1}^{n+1}\alpha_i)}. \tag{1.2}$$

A simple direct proof is given in, for example, [2]. For a historical review of a wider class of integrals, see [3]. Taking linear combinations of these results allows one to integrate polynomials and more general power series in x_i over spheres.

In applications, it may be necessary to use some angular coordinates intrinsic to the sphere. Consider the D-dimensional unit sphere S^D, and let $D = 2n + \epsilon$, with $\epsilon = 0, 1$ according to whether D is even or odd. By introducing plane polar coordinates (μ_i, ϕ_i) for orthogonal 2-planes in $\mathbb{R}^{D+1}$, we have $\lfloor (D+1)/2 \rfloor$ angular coordinates ϕ_i, $i = 1, \dots, n+\epsilon$, with independent periods 2π. The flat metric on $\mathbb{R}^{D+1}$ induces the round metric on S^D given by

$$\mathrm{d}s_D^2 = \sum_{i=1}^{n+1} \mathrm{d}\mu_i^2 + \sum_{i=1}^{n+\epsilon} \mu_i^2\, \mathrm{d}\phi_i^2, \tag{1.3}$$

where μ_i satisfy the constraint

$$\sum_{i=1}^{n+1} \mu_i^2 = 1. \tag{1.4}$$

The metric coefficients are independent of ϕ_i, i.e. $\partial/\partial\phi_i$ are commuting Killing vectors; they represent rotational symmetries. One can imagine situations in which one has to consider functions that are independent of ϕ_i, and so are expressible in terms of μ_i only. These are a generalization to higher dimensions of axisymmetric functions on S^2, which in 3-dimensional spherical polar coordinates depend on $\mu = \cos\theta$ but not the azimuthal coordinate ϕ. This motivates us to consider integrals that are analogous to (1.2), but over S^D and involving powers of μ_i. The main result that we shall prove is that, for $\alpha_j \geq -1$,

$$\int_{S^D} \prod_{j=1}^{n+\epsilon} \mu_j^{\alpha_j} = \frac{2\pi^{(D+1)/2}\prod_{j=1}^{n+\epsilon}\Gamma(1+\frac{1}{2}\alpha_j)}{\Gamma(\frac{D+1}{2}+\frac{1}{2}\sum_{i=1}^{n+\epsilon}\alpha_i)}. \tag{1.5}$$

As an application, we shall evaluate an integral arising in [4], which concerns a correspondence between fluid mechanics on spheres and black holes in AdS (anti-de Sitter) spacetime.

2 Proof of general result

Let X_I, $I = 1, \dots, D+1$ be cartesian coordinates for $\mathbb{R}^{D+1}$. We introduce sets of plane polar coordinates (μ_i, ϕ_i) for the (X_{2i-1}, X_{2i})-planes by

$$(X_{2i-1}, X_{2i}) = (\mu_i \cos\phi_i, \mu_i \sin\phi_i), \tag{2.1}$$

for $i = 1, \ldots, n + \epsilon$. If D is even, then we instead define μ_{n+1} by

$$X_{2n+1} = \mu_{n+1}. \tag{2.2}$$

The coordinates $(\mu_1, \ldots, \mu_{n+1}, \phi_1, \ldots, \phi_{n+\epsilon})$ cover $\mathbb{R}^{D+1}$, with ranges $\mu_i \geq 0$ for $i = 1, \ldots, n+\epsilon$, μ_{n+1} unrestricted if D is even, and $0 \leq \phi_i < 2\pi$ for all i.

The D-dimensional unit sphere $S^D \subset \mathbb{R}^{D+1}$ is the hypersurface $\sum_{I=1}^{D+1} X_I^2 = 1$, on which the round metric is (1.3). Bearing in mind the constraint (1.4), it can be expressed as

$$\mathrm{d}s_D^2 = \mathrm{d}s_n^2 + \sum_{i=1}^{n+\epsilon} \mu_i^2 \, \mathrm{d}\phi_i^2, \tag{2.3}$$

where

$$\mathrm{d}s_n^2 = \sum_{i=1}^{n+1} \mathrm{d}\mu_i^2. \tag{2.4}$$

If we regard μ_i as cartesian coordinates for $\mathbb{R}^{n+1}$, then (2.4) can be interpreted as the round metric on $S^n \subset \mathbb{R}^{n+1}$. A difference is that there no constraints on the signs of μ_i as coordinates for $\mathbb{R}^{n+1}$. On the sphere S^n, we again have the constraint (1.4).

The interpretation of μ_i as either coordinates for $\mathbb{R}^{D+1}$ or for $\mathbb{R}^{n+1}$ enables us to reduce an integral over S^D that is independent of the ϕ_i coordinates to an integral over S^n: we have a "sphere within a sphere". Note that

$$\prod_{l=0}^{n+\epsilon} \int_0^{2\pi} \mathrm{d}\phi_l \int_{\sum_{i=1}^{n+1} \mu_i^2 = 1, \, \mu_1, \ldots, \mu_{n+\epsilon} \geq 0} \mathrm{d}^n \mu \prod_{j=1}^{n+\epsilon} \mu_j^{\alpha_j + 1} = \pi^{n+\epsilon} \int_{\sum_{i=1}^{n+1} \mu_i^2 = 1} \mathrm{d}^n \mu \prod_{j=1}^{n+\epsilon} |\mu_j|^{\alpha_j + 1}, \tag{2.5}$$

because the ϕ_l integrals give a factor of $(2\pi)^{n+\epsilon}$, and removing the sign constraints on $\mu_1, \ldots, \mu_{n+\epsilon}$ gives a factor of $2^{-(n+\epsilon)}$. The meaning of $\mathrm{d}^n\mu$ should be clear. Explicitly, one can, for example, eliminate μ_{n+1} from the integrand in favour of $\mu_1, \ldots, \mu_n$ using the constraint (1.4). Then $\mathrm{d}^n \mu$ means $\prod_{k=1}^n \mathrm{d}\mu_k$, bearing in mind that for each choice of $(\mu_1, \ldots, \mu_n)$ we must account for both signs of μ_{n+1} on the right. Expressing this in terms of integrals over S^D and S^n, with respective metrics (1.3) and (2.4), we have

$$\int_{S^D} \prod_{j=1}^{n+\epsilon} \mu_j^{\alpha_j} = \pi^{n+\epsilon} \int_{S^n} \prod_{j=1}^{n+\epsilon} |\mu_j|^{\alpha_j + 1}. \tag{2.6}$$

Using the Dirichlet integral (1.2) for integration over S^n, remembering for even D that it includes a factor of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we hence obtain our main result (1.5).

3 Application: fluid/gravity correspondence

An explicit application of our main result is to a missing step in [4], which studies the fluid/gravity correspondence. It is argued that there is a duality between a conformal fluid on S^D that solves the relativistic Navier–Stokes equations and a large black hole in AdS_{D+2} that solves the Einstein equations. For one specific example, in arbitrary dimensions, the fluid is uncharged and rigidly rotating, and the black hole is the Kerr–AdS solution, with a horizon radius much larger than the AdS radius. One can compare the thermodynamics of both sides of the correspondence. From the correspondence for non-rotating solutions, one can make predictions for rotating solutions.

On the fluid side of the correspondence, one considers the spacetime

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}s_D^2, \tag{3.1}$$

where $\mathrm{d}s_D^2$ is the round metric on S^D (1.3). The spacetime is filled with a fluid with velocity

$$u^a\partial_a = \gamma\Big(\frac{\partial}{\partial t} + \sum_{i=1}^{n+\epsilon}\omega_i\frac{\partial}{\partial\phi_i}\Big), \tag{3.2}$$

where

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \quad v^2 = \sum_{i=1}^{n+\epsilon}\mu_i^2\omega_i^2, \tag{3.3}$$

and $\omega_i^2 < 1$. One computes the energy-momentum tensor and currents. Integration gives conserved charges, which can be compared with the gravity side of the correspondence. A missing step in [4] is a proof for all D of a certain integral, namely

$$\int_{S^D}\Big(1-\sum_{j=1}^{n+\epsilon}\mu_j^2\omega_j^2\Big)^{-(D+1)/2} = \frac{2\pi^{(D+1)/2}}{\Gamma(\frac{D+1}{2})\prod_{j=1}^{n+\epsilon}(1-\omega_j^2)}. \tag{3.4}$$

Equivalently, we have

$$\int_{S^D}\gamma^{D+1} = \frac{V_D}{\prod_{j=1}^{n+\epsilon}(1-\omega_j^2)}, \quad V_D = \frac{2\pi^{(D+1)/2}}{\Gamma(\frac{D+1}{2})}, \tag{3.5}$$

where V_D is the volume of S^D. Using this result, one then finds agreement between the two sides of the correspondence.

To prove the required integral, we first use binomial expansions to obtain

$$\Big(1-\sum_{j=1}^{n+\epsilon}\mu_j^2\omega_j^2\Big)^{-(D+1)/2} = \sum_{k_1,\ldots,k_{n+\epsilon}\geq 0}(\tfrac{D+1}{2})(\tfrac{D+1}{2}+1)\ldots(\tfrac{D+1}{2}+k-1)\prod_{j=1}^{n+\epsilon}\frac{(\mu_j\omega_j)^{2k_j}}{(k_j)!}, \tag{3.6}$$

where

$$k = \sum_{j=1}^{n+\epsilon}k_j. \tag{3.7}$$

From the main result (1.5), if k_j are non-negative integers, then

$$\int_{S^D}\prod_{j=1}^{n+\epsilon}\mu_j^{2k_j} = \frac{2\pi^{(D+1)/2}\prod_{j=1}^{n+\epsilon}(k_j)!}{\Gamma(\frac{D+1}{2}+k)}. \tag{3.8}$$

Using this and the expansion of $\prod_{j=1}^{n+\epsilon}(1-\omega_j^2)^{-1}$, we hence obtain the integral (3.4).

References

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